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Diversity and Arbitrage in a Regulatory Breakup Model

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Abstract In 1999 Robert Fernholz observed an inconsistency between the normative assumption of existence of an equivalent martingale measure (EMM) and the empirical reality of diversity in equity markets. We explore a method of imposing diversity on market models by a type of antitrust regulation that is compatible with EMMs. The regulatory procedure breaks up companies that become too large, while holding the total number of companies constant by imposing a simultaneous merge of other companies. The regulatory events are assumed to have no impact on portfolio values. As an example, regulation is imposed on a market model in which diversity is maintained via a log-pole in the drift of the largest company. The result is the removal of arbitrage opportunities from this market while maintaining the market's diversity.

Keywords Diversity · Arbitrage · Relative arbitrage · Equivalent martingale measure · Antitrust · Regulation

JEL Classification G11

1 Introduction

What does the empirical phenomenon of diversity in equity markets imply about investment opportunities in those markets? The answer depends on the mechanism by which diversity is maintained.

The notion of diversity, the condition that no company's capitalization (shares multiplied by stock price) may approach that of the entire market, was introduced by Robert Fernholz in the paper Fernholz (1999a) and the book Fernholz (2002) (see also the recent review Fernholz and Karatzas (2009)). He made the observation in Fernholz (1999b) that one of the most useful tools of financial mathematics, the equivalent martingale measure (EMM), implies for a large class of models something grossly inconsistent with real markets: lack of diversity. Historically, the major world stock markets have been diverse, and they should be expected to remain so as long as they are subject to a form of antitrust regulation that prevents concentration of capital into a single company.

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Fernholz demonstrated under common assumptions of financial market modeling that diverse market models necessarily admit strong relative arbitrage with respect to the market portfolio. Portfolio π is a strong relative arbitrage with respect to portfolio ρ on horizon $[0,T]$ if π strictly outperforms ρ at time T with probability one. A sufficient set of assumptions are: capitalizations are modeled by Itô processes that pay no dividends, trading may occur in continuous time with no transaction costs, and the covariance process of the log capitalizations is uniformly elliptic. Importantly, the relative arbitrage portfolios of Fernholz do not depend on the parameters of the market, and therefore do not require estimation of these parameters to construct in practice. They are long-only portfolios (no short sales) derived from portfolio generating functions (see Fernholz (1999b, 2002); Fernholz et al. (2005); Fernholz and Karatzas (2009, 2005)), requiring only the weights of the market portfolio as input. If, additionally, the covariance process is bounded from above uniformly in time, then no equivalent local martingale measure (ELMM) is possible for such models. Therefore the fundamental theorem of asset pricing of Delbaen and Schachermayer (1994) implies that they admit a free lunch with vanishing risk (FLVR).

To make the case that the argument above pertains to the existence of (approximate) relative arbitrages in real markets, dividends must be taken into account. Dividends provide a means for large companies to slow their growth in terms of capitalization while still generating competitive total returns (stock return + dividend return) for their shareholders. An exploratory statistical analysis in Fernholz (1998) of the dividends paid by companies traded on U.S. equity exchanges from 1967-1996 suggests that this factor has historically been insufficient to jeopardize the argument for existence of relative arbitrage with respect to the U.S. market portfolio over this period, before accounting for transaction costs.

It is not easy to formulate diverse Itô process models (however see Osterrieder and Rheinländer (2006) for a clever probabilistic construction utilizing a non-equivalent measure change). Almost all market models commonly used in the literature, including geometric Brownian motion, are not diverse, and therefore do not accurately model reality. Any diverse Itô process model with uniformly elliptic and uniformly bounded covariance must have the characteristic that the difference in the rate of expected return of the largest company, compared to some other company, diverges to $-\infty$ as the largest approaches a relative size cap (see Fernholz et al. (2005)). Some possible economic rationale to support this type of model includes: difficulties in achieving high return on investment for very largely capitalized companies and the cost of antitrust suits brought against such companies.

Since the onset of antitrust regulation in the U.S. in the late 19th century, there have been two main regulatory methods of dealing with companies which get too large: antitrust suits or fines, and antitrust breakup. The latter is rarely used, with some notable examples being the breakups of Standard Oil (1911) Standard Oil v United States (1911) and AT&T (1982) United States v AT&T (1982). Suits or fines are used much more often than breakups to discipline companies that are deemed to be dominating their market in an unfair manner. Recent examples in Europe include Microsoft in Microsoft Corp. v European Commission (2004) in 2004 (€497 million) and Intel in European Commission v Intel (2009) in 2009 (€1.06 billion), both being fined by the European Union for anticompetitive practices. Models in which diversity is maintained via the rate of expected return of any company diverging to $-\infty$ as that company's relative size becomes very large can be interpreted as continuous-path approximations of the case where suits or fines are used to regulate big companies. Models in which regulatory breakup is the primary means of maintaining market diversity have not been well-studied from a mathematical point of view in the financial mathematics literature. They are the subject of this paper.

When a company is fined money, this directly and adversely affects the value of the company, so the risk of antitrust fines is a mark against investing in large companies. In contrast, the key mathematical feature of a corporate breakup with regards to investment is that capital need not be removed from the system. That is, when a company is broken into parts, no net value needs to be lost. Indeed, from a regulator's perspective,

avoidance of monopolies maintains the viability of an industry's innovation and growth prospects. Although it need not be the case in practice, for simplicity, we make the modeling assumption that total market values of companies, as well as the portfolio values of investors, are *conserved* at each regulatory breakup. The conservation of portfolio value implies that the capital gains process from investment in equity is not the stochastic integral of the trading strategy (shares of equity) with respect to the stock capitalization process. Instead, a net capitalization process, with the finite number of regulatory jumps removed, plays the role of integrator.

Another assumption we make is that the number of companies remains constant. This may seem inconsistent with the breakup of companies, but in our typical example of regulation we balance the number of companies in the economy by also requiring that two companies merge into a new company at the same time as regulation splits a company into two. This is imposed mainly for mathematical simplicity. It isolates the effect of regulation on diversity and arbitrage while working in the familiar context of $\mathbb{R}^n$-valued Itô processes.

As an application we examine a regulated form of a log-pole market model, a diverse model admitting relative arbitrage with respect to the market portfolio. The regulation procedure removes the arbitrage opportunities from the market, resulting in a diverse and arbitrage-free market. Furthermore, the regulated form satisfies the notion of "sufficient intrinsic volatility" of the market, a more general sufficient condition for relative arbitrage in unregulated models (see Fernholz and Karatzas (2005)). These results do not contradict the work of Fernholz et al., because in our model it is the regulated capitalization process that is diverse and the net capitalization process (which has regulatory jumps removed) that has an EMM.

This paper is organized as follows. Section 2 defines the class of premodels for the regulation procedure, the admissible trading strategies, portfolios, and the notion of diversity. In Section 3 we introduce the regulatory procedure, including defining the regulatory mapping and the triggering mechanism for regulation. Our exemplar of regulation, the split-merge rule, is also introduced, which essentially splits the biggest company and forces the smallest two companies to merge at a regulatory event. The issue of arbitrage in regulated markets is thoroughly explored and compared to the results of Fernholz et al. regarding arbitrage and diversity in unregulated mode. Section 4 applies the regulatory procedure to geometric Brownian motion and to a log-pole market model to illustrate the compatibility of diversity and EMMs in regulated models. Section 5 presents some concluding remarks and directions for future research. Section 6 contains several proofs.

2 Premodel

We first introduce the class of models that we will consider for regulation. We also define the set of trading strategies that are admissible for discussions of arbitrage and define the notion of a portfolio for discussions of relative arbitrage.

The stock capitalization process $\widetilde{X} = (\widetilde{X}_{1,t}, \dots, \widetilde{X}_{n,t})'_{t\geq 0}$ represents the capitalizations (number of shares multiplied by stock price) of the $n \geq 2$ companies which are traded on an exchange, where the notation A' denotes the transpose of the matrix A. The stock capitalizations are each assumed to be almost surely (a.s.) strictly positive for all time, with $\widetilde{X}$ taking values in the open, connected, conic set $O^x \subseteq \mathbb{R}^n_{++} := (0,\infty)^n$. The dynamics of $\widetilde{X}$ is determined by the stochastic differential equation (SDE)

$$d\widetilde{X}_{i,t} = \widetilde{X}_{i,t}\left(b_i(\widetilde{X}_t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(\widetilde{X}_t)dW_{\nu,t}\right), \quad 1 \leq i \leq n, \tag{2.1}$$

$$\widetilde{X}_0 = x_0 \in O^x, \tag{2.2}$$

for which $(\Omega,\mathscr{F},\mathbb{F},\widetilde{X},W,P)$ is a solution, where W is a d-dimensional Brownian motion with $d \geq n$. The functions $b(\cdot)$ and $\sigma(\cdot)$ are assumed to be locally bounded Borel functions. We require that the SDE (2.1) satisfies strong existence and pathwise uniqueness for any initial $x_0 \in O^x$, and that $P(\widetilde{X}_t \in O^x,\ \forall t \geq 0) = 1$. We shall only consider volatility matrices $\sigma(x) \in \mathbb{R}^{n\times d}$ having full rank n, $\forall x \in O^x$, which guarantees that no stock's risk can be completely hedged over any time interval by investment in the other stocks. We assume that $\mathscr{F}$ and $\mathscr{F}_0$ contain $\mathscr{N}$, the P-null sets, and consider only the case where the filtration is the augmented Brownian filtration $\mathbb{F} = \mathbb{F}^W := \{\mathscr{F}_t^W\}_{0\leq t<\infty}$, where $\mathscr{F}_t^W := \sigma\big(\{W_s\}_{0\leq s\leq t}\big) \bigvee \mathscr{N}$.

The process B represents a money market account, for which we impose that a.s. $B \equiv 1$, corresponding to zero interest rate. Other standing assumptions are that capitalizations are exogenously determined, no dividends are paid, markets are perfectly liquid, trading is frictionless (no transaction costs) and may occur in arbitrary quantities, and there are no taxes.

2.1 Investment in the Premodel

The model for investment in the risky assets of the premodel is of the usual type for equity market models. A *trading strategy* $\widetilde{H}_t' := (\widetilde{H}_{1,t},\ldots,\widetilde{H}_{n,t})$ is a predictable process representing the number of shares held of each stock. Note that since $\widetilde{X}$ is a stock capitalization process, the number of shares outstanding of each company has effectively been normalized to one, and so $\widetilde{H}$ is with respect to this one share. The *wealth process* $V^{w,\widetilde{H}}$ associated to trading strategy $\widetilde{H}$ is assumed to be *self-financing*, so satisfies

$$\widetilde{V}_t^{w,\widetilde{H}} = \widetilde{H}_t^B + \widetilde{H}_t'\widetilde{X}_t = w + (\widetilde{H}\cdot\widetilde{X})_t,$$

where w is the initial wealth and $\widetilde{H}^B$ is the number of shares of money market account. We follow Delbaen and Schachermayer's definition of admissible trading strategies from Delbaen and Schachermayer (2006).

Definition 2.1 *Admissible trading strategies* are predictable processes $\widetilde{H}$ such that

(i) $\widetilde{H}$ is $\widetilde{X}$-integrable, that is, the stochastic integral $\widetilde{H}\cdot\widetilde{X} = \big(\int_0^t \widetilde{H}_s d\widetilde{X}_s\big)_{t\geq 0}$ is well-defined in the sense of stochastic integration theory for semimartingales.

(ii) There is a constant R such that a.s.

$$(\widetilde{H}\cdot\widetilde{X})_t \geq -R, \quad \forall t \geq 0. \tag{2.3}$$

The second restriction is designed to rule out "doubling strategies" (see Karatzas and Shreve (1998), p.8) and represent the realistic constraint that credit lines are limited.

It will also be useful in the context of relative arbitrage to develop the notion of a portfolio, *à la* Fernholz and Karatzas (2009).

Definition 2.2 A *portfolio* $\widetilde{\pi}$ is an $\mathbb{F}$-progressively measurable n-dimensional process bounded uniformly in (t,ω), with values in the set

$$\bigcup_{\kappa\in\mathbb{N}} \big\{(\pi_1,\ldots\pi_n) \in \mathbb{R}^n \mid \pi_1^2+\ldots+\pi_n^2 \leq \kappa^2,\ \sum_{i=1}^n \pi_i = 1\big\}. \tag{2.4}$$

A *long-only portfolio* $\widetilde{\pi}$ is a portfolio that takes values in the unit simplex

$$\Delta^n := \big\{(\pi_1,\ldots\pi_n) \in \mathbb{R}^n \mid \pi_1 \geq 0,\ldots\pi_n \geq 0,\ \sum_{i=1}^n \pi_i = 1\big\}.$$

A portfolio $\widetilde{\pi}$ represents the fractional amount of an investor's wealth invested in each stock. In contrast to a trading strategy, no borrowing from or lending to the money market is allowed when investment occurs via a portfolio. This requirement may be dropped and (2.4) may be relaxed in favor of more general integrability conditions, for example see Fernholz and Karatzas (2010). However for our purposes here, these restrictions suffice.

For $w \in \mathbb{R}_{++}$, the wealth process $\widetilde{V}^{w,\widetilde{\pi}}$ corresponding to a portfolio is defined to be the solution to

$$\begin{aligned} d\widetilde{V}_t^{w,\widetilde{\pi}} &= \widetilde{V}_t^{w,\widetilde{\pi}} \sum_{i=1}^{n} \widetilde{\pi}_{i,t} \frac{d\widetilde{X}_{i,t}}{\widetilde{X}_{i,t}}, \\ &= (\widetilde{V}_t^{w,\widetilde{\pi}})\widetilde{\pi}_t' \left[b(\widetilde{X}_t)dt + \sigma(\widetilde{X}_t)d\widetilde{W}_t \right], \end{aligned} \tag{2.5}$$

which by use of Itô's formula can be verified to be

$$\widetilde{V}_t^{w,\widetilde{\pi}} = w \exp\left\{ \int_0^t \widetilde{\gamma}_{\widetilde{\pi},s} ds + \int_0^t \widetilde{\pi}_s' \sigma(\widetilde{X}_s) d\widetilde{W}_s \right\}, \quad \forall t \geq 0, \tag{2.6}$$

where

$$\widetilde{\gamma}_{\widetilde{\pi}} := \widetilde{\pi}' b(\widetilde{X}) - \frac{1}{2}\widetilde{\pi}' a(\widetilde{X})\widetilde{\pi} \qquad \text{and} \qquad a(\cdot) := \sigma(\cdot)\sigma'(\cdot).$$

The process $\widetilde{\gamma}_{\widetilde{\pi}}$ is called the *growth rate* of the portfolio $\widetilde{\pi}$, and $a(\widetilde{X})$ is called the *covariance process*. See Fernholz and Karatzas (2009) for more details on the properties of these processes.

The definitions of the wealth process $\widetilde{V}^{w,\widetilde{\pi}}$ corresponding to a portfolio and $\widetilde{V}^{w,\widetilde{H}}$ corresponding to a trading strategy are consistent in the sense that any portfolio has an a.s. unique corresponding admissible trading strategy yielding the same wealth process from the same initial wealth. The corresponding trading strategy $\widetilde{H}^{w,\widetilde{\pi}}$ can be obtained from

$$\widetilde{H}_i^{w,\widetilde{\pi}} = \frac{\widetilde{\pi}_i \widetilde{V}^{w,\widetilde{\pi}}}{\widetilde{X}_i}, \quad 1 \leq i \leq n. \tag{2.7}$$

The *market portfolio* $\widetilde{\mu}$ is of particular interest since "beating the market" is often a desirable goal for investors. The market portfolio is simply the relative capitalization of each company in the market with respect to the total:

$$\widetilde{\mu}_{i,t} := \mu_i(\widetilde{X}_t) := \frac{\widetilde{X}_{i,t}}{\sum_{j=1}^n \widetilde{X}_{j,t}}, \quad 1 \leq i \leq n.$$

The market portfolio is a passive portfolio, meaning that once the initial portfolio $\widetilde{\mu}$ is setup, it is not traded. The wealth process $\widetilde{V}_t^{w,\widetilde{\mu}}$ is proportional to the total capitalization of the market, as seen by

$$\widetilde{V}_t^{w,\widetilde{\mu}} = \left(\frac{w}{\sum_{j=1}^n \widetilde{X}_{j,0}} \right) \sum_{j=1}^n \widetilde{X}_{j,t}, \quad \forall t \geq 0.$$

Since the stock capitalization process $\widetilde{X}$ a.s. takes values in $O^x \subseteq \mathbb{R}_{++}^n$, then for $O^\mu := \mu(O^x)$, we have that a.s., $\forall t \geq 0$,

$$\widetilde{\mu}_t \in O^\mu \subseteq \mu(\mathbb{R}_{++}^n) = \Delta_+^n := \left\{ (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n \mid \pi_1 > 0, \ldots, \pi_n > 0, \sum_i^n \pi_i = 1 \right\}.$$

The closure of a set $A \subseteq \mathbb{R}_{++}^n$ will be referred to as $\overline{A}$ and, unless otherwise stated, is taken with respect to the subspace topology of $\mathbb{R}_{++}^n$, and similarly for subsets of Δ_+^n. For example, $\overline{\mathbb{R}_{++}^n} = \mathbb{R}_{++}^n$ and $\overline{\Delta_+^n} = \Delta_+^n$.

2.2 Diversity

The notion of diversity entails that no company may ever become too big in terms of relative capitalization. For generalizations to this notion and their implications see Fernholz et al. (2005). Diversity is a realistic criterion for a market model to satisfy, since it has held empirically in developed equity markets over time and should be expected to continue to hold as long as antitrust regulation prevents capital from concentrating in a single company. In discussions of diversity it is useful to adopt the reverse-order-statistics notation. That is, for $x \in \mathbb{R}^n$,

$$x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}.$$

Definition 2.3 A premodel is *diverse* on $[0,T]$ if there exists $\delta \in (0,1)$ such that a.s.

$$\widetilde{\mu}_{(1),t} < 1-\delta, \qquad \forall \quad 0 \leq t \leq T.$$

A premodel is *weakly diverse* on $[0,T]$ if there exists $\delta \in (0,1)$ such that

$$\frac{1}{T}\int_0^T \widetilde{\mu}_{(1),t} dt < 1-\delta, \quad \text{a.s.}$$

We will not make much use of diversity until later on, but it is good to keep the definition in mind when considering the regulation procedure proposed herein.

3 Regulated Market Models

3.1 Overview and Modeling Assumptions

The notion of regulation we introduce consists of confining the market weights (except at exit times) in an open set U^μ by a regulatory procedure that

- conserves the number of companies in the market;
- conserves total market capital;
- conserves portfolio wealth;
- causes a jump in company capitalizations.

Upon exit from U^μ, the market weights are mapped back into U^μ by a deterministic mapping $\mathfrak{R}^\mu$ applied to $\widetilde{\mu}$ at its exit point. Then μ diffuses according to the SDE (2.1) until it exits from U^μ again. The cycle of diffusion and regulation continues on indefinitely, determining a regulated market weight process μ. This idea will be made precise in the following subsection.

The economic motivation behind the regulated market models presented in this paper is to study markets with the feature that companies may merge and split, possibly forced to do so by a regulator, with an aim to explore the ramifications for diversity and arbitrage in these markets. In order to avoid what the authors believe to be unnecessary mathematical complications in the study of these notions, we require that splits and merges only occur simultaneously and in pairs, so that the number of companies in the economy remains a constant. For example, the biggest company may split into two, and simultaneously the smallest two merge into one.

Two crucial assumptions of our regulated market models are that total market capital and portfolio wealth are conserved at each regulation event. These assumptions are indeed idealizations, but the authors believe that the former is a reasonable starting point for studying splits and merges imposed into otherwise continuous path premodels, while the latter is then sensible in consideration of the below remark.

Remark 3.1 Consider a company being split into two smaller companies with capitalization fractions ρ and $1-\rho$ relative to the parent company. If each investor's money in the parent company is also broken up so that they are left with fraction ρ invested in the first offspring and $1-\rho$ invested in the second offspring immediately following the split, then individual portfolio wealth and total market capital are conserved. This mapping of portfolio wealth does not impose any constraints on the trading strategies or portfolios available in the regulated market. Since trading occurs in continuous time, any investor may simply rearrange all of her money just after regulation. That our investor may do this without affecting market prices reflects the assumption that stock capitalizations are exogenously determined, that is, our investor is small relative to the market, and her behavior has negligible impact on asset prices.

The alternative to the wealth conservation assumption would be to impose a random jump in portfolio wealth at regulation. This would be compelling for studies of event-driven arbitrage, but since here our purpose is to study the structural-type arbitrage arising from diversity, we feel that this is a reasonable omission.

3.2 Regulated Markets

In this section we will construct the regulated stock process by means of induction via the diffusion-regulation cycle outlined in the previous subsection. Since the SDE (2.1) for $\widetilde{X}$ satisfies strong existence and pathwise uniqueness, then we need not pass to a new probability space to construct the regulated model. Extensions are possible when (2.1) merely satisfies weak existence and weak uniqueness, but for simplicity of presentation, we do not pursue these generalizations here.

Definition 3.2 A *regulation rule* $\mathfrak{R}^\mu$ with respect to the open, nonempty *regulatory set* $U^\mu \subseteq O^\mu \subseteq \Delta_+^n$ is a Borel function

$$\mathfrak{R}^\mu : \partial U^\mu \to U^\mu .$$

The regulation rule $(U^\mu, \mathfrak{R}^\mu)$ uniquely determines the following set and capital-conserving map of stock capitalizations:

$$\begin{aligned} U^x &:= \mu^{-1}(U^\mu) \subseteq O^x, \\ \mathfrak{R}^x &: \partial U^x \to U^x, \\ \mathfrak{R}^x(x) &:= \left(\sum_{i=1}^n x_i\right) \mathfrak{R}^\mu(\mu(x)). \end{aligned}$$

The inclusion $U^x \subseteq O^x$ follows from our assumption that O^x is conic, which implies $O^x = \mu^{-1}(O^\mu)$. The set U^x is conic, that is $x \in U^x \Rightarrow \lambda x \in U^x$, $\forall \lambda > 0$, allowing any total market value for a given $\mu \in U^\mu$. Therefore, the market capitalization M is a degree of freedom for the regulatory mapping, in the sense that $\mu(\mathfrak{R}^x(x)) = \mu(\mathfrak{R}^x(\lambda x))$, $\forall \lambda > 0$. Specification of $(U^x, \mathfrak{R}^x)$ or $(U^\mu, \mathfrak{R}^\mu)$ uniquely determines the other, so we refer to either as "regulation rules," and in discussion drop the labels and refer to them as $(U, \mathfrak{R})$.

Define the following processes and random variables:

$$\begin{aligned} W^1 &:= W, & X^1 &:= \widetilde{X}, \\ \tau_0 &:= 0, & \tau_1 &:= \varsigma_1 := \inf\left\{t > 0 \mid \mu(X_t^1) \notin U^\mu\right\}. \end{aligned}$$

The process X^1 will serve as the first piece of the regulated capitalization process on the stochastic interval $[0, \tau_1] := \{(t, \omega) \in [0, \infty) \times \Omega \mid 0 \le t \le \tau_1(\omega)\}$. At τ_1, X^1 has just exited U^x, so the regulation procedure

maps the capitalization process to $\mathfrak{R}^x(X^1_{\varsigma_1})$, and the regulated capitalization process continues from that point according to the dynamics given by the SDE (2.1). To implement this, define the following variables and processes inductively, $\forall k \in \mathbb{N}$ on $\{\tau_{k-1} < \infty\}$, terminating if $P(\tau_{k-1} < \infty) = 0$:

$$
\begin{aligned}
W^k_t &:= W_{\tau_{k-1}+t} - W_{\tau_{k-1}}, \quad \forall t \geq 0, \\
dX^k_{i,t} &= X^k_{i,t}\left(b_i(X^k_t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(X^k_t)dW^k_{\nu,t}\right), \quad 1 \leq i \leq n, \\
X^k_0 &= \begin{cases} y_0 \in U^x, & \text{for } k = 1, \\ \mathfrak{R}^x(X^{k-1}_{\varsigma_{k-1}}), & \text{for } k > 1, \end{cases} \\
\varsigma_k &:= \inf\left\{t > 0 \mid X^k_t \notin U^x\right\}, \\
\tau_k &:= \sum_{j=1}^{k} \varsigma_j.
\end{aligned} \tag{3.1}
$$

If for some $k \in \mathbb{N}$ the induction terminates because $P(\tau_{k-1} < \infty) = 0$, then on $\{\tau_{k-1} < \infty\}$, $\forall m \geq k$ define $X^m \equiv y_0 \in U^x$, $\tau_m = \infty$, $\varsigma_m = 0$. Use these same definitions $\forall m \in \mathbb{N}$ on $\{\tau_{m-1} = \infty\}$. These cases are included for completeness and their specifics are irrelevant for the subsequent development.

By the strong Markov property of Brownian motion and stationarity of its increments, if $P(\tau_{k-1} < \infty) > 0$, then for

$$\mathscr{F}^k_t := \mathscr{F}_{\tau_{k-1}+t}, \qquad \mathbb{F}^k := \{\mathscr{F}^k_t\}_{t \geq 0},$$

$(W^k, \mathbb{F}^k)$ is a Brownian motion on $\{\tau_{k-1} < \infty\}$, that is, on $(\Omega \cap \{\tau_{k-1} < \infty\}, \mathscr{F} \cap \{\tau_{k-1} < \infty\})$. The SDE (3.1) for $k \geq 2$ has the same form as the SDE (2.1) for $\widetilde{X}$, but with W^k in place of W, and with initial condition $X^k_0 = \mathfrak{R}^x(X^k_{\varsigma_{k-1}})$ a.s. on $\{\tau_{k-1} < \infty\}$. Therefore, on $\{\tau_{k-1} < \infty\}$ by strong existence, there exists X^k adapted to $\mathbb{F}^k$ satisfying (3.1).

Each ς_k is a stopping time with respect to $\mathbb{F}^k$, since it is the hitting time of the closed set $\mathbb{R}^n \setminus U^x$ by the continuous process X^k. Therefore each τ_k is an $\mathbb{F}$-stopping time. Since $\mathfrak{R}^x(X^{k-1}_{\varsigma_{k-1}}) \in U^x$, then $\varsigma_k > 0$ and $\tau_k > \tau_{k-1}$ both a.s. on $\{\tau_{k-1} < \infty\}$, for all k such that $P(\tau_{k-1} < \infty) > 0$. Note that under this construction there is the possibility of explosion, that is, of $\tau_\infty := \lim_{k\to\infty} \tau_k < \infty$. This will be considered in greater detail later on.

We are now ready to define the regulated capitalization process Y by pasting together and shifting the $\{X^k\}_1^\infty$ at the $\{\tau_k\}_1^\infty$ as follows.

Definition 3.3 With respect to regulation rule $(U, \mathfrak{R})$ and initial point $y_0 \in U^x$, the *regulated capitalization process* is defined as

$$
Y_t(\omega) := \begin{cases} X^1_0 \mathbf{1}_{\{0\}}(t) + \sum_{k=1}^\infty \mathbf{1}_{(\tau_{k-1},\tau_k]}(t,\omega) X^k_{t-\tau_{k-1}}(\omega), & \forall (t,\omega) \in [0,\tau_\infty), \\ X^1_0, & \forall (t,\omega) \notin [0,\tau_\infty), \end{cases} \tag{3.2}
$$

where $P(X^1_0 = y_0) = 1$. If $P(\tau_\infty = \infty) = 1$, then we call the triple $(y_0, U, \mathfrak{R})$ *viable* for the premodel.

To count the number of regulations by time t, let $N_t := \sum_{k=1}^\infty \mathbf{1}_{\{t > \tau_k\}}$, $t \geq 0$. Since each τ_k is a stopping time, N is $\mathbb{F}$-adapted. Each X^k is $\mathbb{F}^k$-progressive, so by a standard shift argument Y can be seen to be $\mathbb{F}$-adapted, and therefore also $\mathbb{F}$-progressive due to the left-continuity of its paths.

3.3 Investment in the Regulated Market

As remarked earlier, in the regulated model wealth is unaltered by a regulatory event. Specifically, the wealth process $V^{w,H}$ of trading strategy H does not jump upon redistribution of market capital at τ_k^+. However, the capitalization Y does jump. This implies that the capital gains of a trading strategy can't be the stochastic integral of the trading strategy with respect to the regulated capitalization process. In order to recover the useful tool of representing the capital gains process as a stochastic integral, we define a net capitalization process $\widehat{Y}$, which only accounts for the non-regulatory movements of Y.

Definition 3.4 The *net capitalization process* $\widehat{Y}$ is defined as

$$\widehat{Y}_t := \begin{cases} Y_t - \sum_{k=1}^{N_t} (\mathfrak{R}^x(Y^k_{\tau_k}) - Y_{\tau_k}), & \forall (t,\omega) \in [0,\tau_\infty), \\ Y_0, & \forall (t,\omega) \notin [0,\tau_\infty). \end{cases} \tag{3.3}$$

The process $\widehat{Y}$ is $\mathbb{F}$-adapted since Y and N are adapted. If the regulated market is viable, then a.s. $\widehat{Y}$ has continuous paths since then a.s. Y has piecewise continuous paths, jumping only at the τ_k. The following representations of $\widehat{Y}$ will also be useful and are obtainable from the definitions of Y and $\widehat{Y}$ and (3.1).

$$\widehat{Y}_t = Y_0 + \sum_{k=1}^{N_t+1} (X^k_{(t-\tau_{k-1})\wedge\varsigma_k} - X^k_0), \quad \forall (\omega,t) \in [0,\tau_\infty) \tag{3.4}$$

$$d\widehat{Y}_{i,t} = Y_{i,t}\left[b_i(Y_t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(Y_t)dW_t\right], \quad 1 \le i \le n, \quad \text{on } [0,\tau_\infty). \tag{3.5}$$

The net capitalization process is the correct process to fulfill the role of integrator for a trading strategy in the regulated market. To see this, let trading strategy H be the number of shares invested in the regulated stock process Y. The wealth process V^H should be locally self-financing on each stochastic interval $(\tau_{k-1},\tau_k]$ and without jumps, so that

$$V_t^{w,H} = w + \sum_{k=1}^{N_t} \int_{\tau_{k-1}^+}^{\tau_k} H_s dY_s + \int_{\tau_{N_t}^+}^{t} H_s dY_s.$$

Then since $\widehat{Y}_t - \widehat{Y}_{\tau_{k-1}} = Y_t - Y_{\tau_{k-1}}$ on $(\tau_{k-1},\tau_k]$ for all $k \in \mathbb{N}$, we have

$$\begin{aligned} V_t^{w,H} &= w + \sum_{k=1}^{N_t} \int_{\tau_{k-1}}^{\tau_k} H_s d\widehat{Y}_s + \int_{\tau_{N_t}}^{t} H_s d\widehat{Y}_s, \\ &= w + (H \cdot \widehat{Y})_t. \end{aligned}$$

This motivates the following natural analog of the usual self-financing condition.

Definition 3.5 In a viable regulated market, a wealth process $V^{w,H}$ corresponding to $\widehat{Y}$-integrable trading strategy H is called *self-financing* in the regulated market if

$$V_t^{w,H} = w + (H \cdot \widehat{Y})_t, \quad \forall t \ge 0.$$

As in the premodel, in a viable regulated model we will henceforth assume that all wealth processes are self-financing and that all trading strategies are $\widehat{Y}$-admissible, which means that H is $\widehat{Y}$-integrable, and $H \cdot \widehat{Y}$ is a.s. bounded from below uniformly in time, paralleling Definition 2.1.

A portfolio in the regulated model will be denoted by π, and is a process meeting the requirements of Definition 2.2. It represents the fractional amount of total wealth invested in the regulated stocks Y. Paralleling the premodel (2.5) for initial wealth $w \in \mathbb{R}_{++}$, the wealth process $V^{w,\pi}$ corresponding to π is given by

$$V_t^{w,\pi} = w \exp\left\{ \int_0^t \gamma_{\pi,s} ds + \int_0^t \pi_s' \sigma(Y_s) dW_s \right\}, \quad \forall t \geq 0, \tag{3.6}$$

where

$$\gamma_\pi := \pi' b(Y) - \frac{1}{2} \pi' a(Y) \pi, \qquad a(\cdot) = \sigma(\cdot)\sigma'(\cdot).$$

The market portfolio is the portfolio with the same weights μ as the market. Note that unlike $\widetilde{H}^{w,\widetilde{\mu}}$, which is constant, $H^{w,\mu}$ is piecewise constant, jumping at the τ_k. All portfolios, including the market portfolio, have wealth processes of identical functional form (compare (3.6) and (2.6)) in the regulated model and in the premodel. Therefore, from a mathematical viewpoint, the differences in investment opportunities in these markets are completely due to the differences in dynamics of $b(Y)$ and $\sigma(Y)$ compared to $b(\widetilde{X})$ and $\sigma(\widetilde{X})$, which is in turn due to confining $\widetilde{X}$ to U^x via $\mathfrak{R}^x$ to obtain Y.

3.4 Split-Merge Regulation

The exemplar for regulation used in this paper is the split-merge regulation rule. The basic economic motivation behind split-merge regulation is that it provides a means for regulators to control the size of the largest company in the economy. At each τ_k, the largest company is split into two new companies of equal capitalization. In order to avoid the mathematical complications of a market model with a variable number of companies, we also impose that at each τ_k the smallest two companies merge, so that the total number of companies is a constant, n. A natural trigger for when regulators might force a large company to split is company size. For example, regulation may be triggered when the biggest company reaches $1-\delta$ in relative capitalization.

The purpose of this subsection is to define the class of split-merge regulation rules and to find sufficient conditions for the viability of this class. These results are summarized in Lemma 3.10.

To identify which company by index occupies the kth rank at time t, we use the random function $p_t(\cdot)$ so that $\mu_{p_t(k),t} = \mu_{(k),t}$, for $1 \leq k \leq n$. Similarly, for the vector $x := (x_1, \ldots, x_n)$ we use $p(\cdot)$ satisfying $x_{p(k)} = x_{(k)}$, for $1 \leq k \leq n$. In the event that several components are tied, for example $x_{(k)} = \ldots x_{(k+j)}$, then ties are settled by $p(k) < \ldots < p(k+j)$.

To define the notion of split-merge regulation, we first define a regulation prerule, which captures the essential idea but still requires some technical refinement.

Definition 3.6 In a market where $n \geq 3$, a *split-merge regulation prerule* $(U^\mu, \check{\mathfrak{R}}^\mu)$ with respect to open, nonempty regulatory set $U^\mu \subseteq O^\mu$ is a mapping

$$\check{\mathfrak{R}}^\mu : \bar{U}^\mu \to \Delta_+^n$$

such that

$$\check{\mathfrak{R}}^\mu(\mu) = \mu, \quad \forall \mu \in U^\mu,$$

and $\check{\mathfrak{R}}^{\mu} \restriction_{\partial U^{\mu}}$ is specified by the map:

$$\begin{aligned} \mu_{p(1)} &\mapsto \frac{\mu_{p(1)}}{2}, \\ \mu_{p(n-1)} &\mapsto \frac{\mu_{p(1)}}{2}, \\ \mu_{p(n)} &\mapsto \mu_{p(n)} + \mu_{p(n-1)}, \\ \mu_{p(k)} &\mapsto \mu_{p(k)}, \quad \forall k : 2 \leq k < n-1. \end{aligned}$$

The split-merge regulation prerule can be interpreted as splitting the largest company in half into two new companies and forcing the smallest two companies to merge into a new company. The condition $n \geq 3$ insures that these companies are distinct. The new companies from the split are assigned the indices of the previous largest and the previous second smallest companies. The new company from the merge is assigned the index of the previous smallest company.

Remark 3.7 Due to the interchange of indices, this interpretation makes economic sense only in a market model where the companies are taken to be generic, that is, they have no firm-specific (index-specific) properties. For example, in a market model where sector-specific correlations are being modeled, it would not make sense for an oil company resulting from a split to take over the index of a technology company freed up from a merge, since the subsequent correlations would not be realistic. The examples in this paper focus on generic market models, so this interpretation is sensible for them.

A split-merge regulation prerule $(U^{\mu}, \check{\mathfrak{R}}^{\mu})$ is not quite suitable for our notion of split-merge regulation, because in the event that $\mu_{(1),\tau_k} = \mu_{(2),\tau_k} = \ldots = \mu_{(j),\tau_k}$, we desire that all of these largest companies be broken up, not just one of them. This can be easily accomplished, however, by repeating the procedure n times.

Definition 3.8 If $n \geq 3$ and split-merge regulation prerule $(U^{\mu}, \check{\mathfrak{R}}^{\mu})$ is into $\bar{U}^{\mu}$, then we may define

$$\mathfrak{R}^{\mu} := (\underbrace{\check{\mathfrak{R}}^{\mu} \circ \ldots \circ \check{\mathfrak{R}}^{\mu}}_{n \text{ compositions}}) \restriction_{\partial U^{\mu}} .$$

If $\mathfrak{R}^{\mu}$ is into U^{μ}, then we may restrict the codomain to U^{μ}, and we call the resulting function $(U^{\mu}, \mathfrak{R}^{\mu})$ the *split-merge regulation rule* associated with $(U^{\mu}, \check{\mathfrak{R}}^{\mu})$.

Note that the above definition implies that when a split-merge regulation rule exists, it is a regulation rule. The following technical lemma will be handy for verifying the viability of split-merge rules. We use the notation $C_b^2(\Delta_+^n, \mathbb{R})$ to denote the continuous bounded functions from Δ_+^n to $\mathbb{R}$ with partial derivatives continuous and bounded through 2nd order.

Lemma 3.9 *If the SDE (2.1) has drift $b(\cdot)$ and volatility $\sigma(\cdot)$ functions which are bounded on U^x, and there exists a function $G \in C_b^2(\Delta_+^n, \mathbb{R})$ such that the regulation rule $(U, \mathfrak{R})$ satisfies either*

$$\begin{aligned} &\inf\{G(\mathfrak{R}^{\mu}(\mu)) - G(\mu) \mid \mu \in \partial U^{\mu}\} > 0 \\ \textit{or} \qquad &\sup\{G(\mathfrak{R}^{\mu}(\mu)) - G(\mu) \mid \mu \in \partial U^{\mu}\} < 0, \end{aligned}$$

where ∂U^{μ} is the boundary of the set U^{μ} taken as a subset of the space Δ_+^n, then the regulated market is viable.

Proof See Section 6.

We turn now to the question of identifying suitable regulatory sets U for split-merge regulation that are both economically compelling and generate viable split-merge rules.

Lemma 3.10 *Suppose the following hold:*

(i) $n \geq 3$.
(ii) $\delta \in (0, \frac{n-1}{n+1})$.
(iii) *The regulatory set,*

$$U^\mu := \{\mu \in \Delta_+^n \mid \mu_{(1)} < 1 - \delta\},$$

satisfies $U^\mu \subseteq O^\mu$.
(iv) $(U^\mu, \check{\mathfrak{R}}^\mu)$ *is a split-merge regulation prerule.*
(v) *The functions* $b(\cdot)$ *and* $\sigma(\cdot)$ *are bounded on* U^x.

Then the split-merge rule $(U^\mu, \mathfrak{R}^\mu)$ *associated with* $(U^\mu, \check{\mathfrak{R}}^\mu)$ *exists and is viable.*

Proof See Section 6.

3.5 Arbitrage

We begin with the notions of arbitrage, relative arbitrage, and no free lunch with vanishing risk (NFLVR). Then we recall the fundamental theorem of asset pricing (FTAP) for locally bounded semimartingales and discuss its implications for regulated market models.

Definition 3.11 In the premodel an *arbitrage* over $[0,T]$ is an admissible trading strategy $\widetilde{H}$ such that

$$P[(\widetilde{H} \cdot \widetilde{X})_T \geq 0] = 1 \quad \text{and} \quad P[(\widetilde{H} \cdot \widetilde{X})_T > 0] > 0. \tag{3.7}$$

A *relative arbitrage* over $[0,T]$ with respect to portfolio $\widetilde{\eta}$ is a portfolio $\widetilde{\pi}$ such that

$$P(\widetilde{V}_T^{1,\widetilde{\pi}} \geq \widetilde{V}_T^{1,\widetilde{\eta}}) = 1 \quad \text{and} \quad P(\widetilde{V}_T^{1,\widetilde{\pi}} > \widetilde{V}_T^{1,\widetilde{\eta}}) > 0. \tag{3.8}$$

The corresponding notions of *strong arbitrage* and *strong relative arbitrage* are defined by making the first inequalities of (3.7) and (3.8) strict, respectively.

The condition NFLVR is a strengthening of the no arbitrage condition, roughly implying that not only are there no arbitrages, but no “approximate arbitrages.” See Delbaen and Schachermayer (2006, 1994, 1998) for a complete exposition.

Definition 3.12 For $T \in \mathbb{R}_{++}$ define

$$\widetilde{K} := \left\{ (\widetilde{H} \cdot \widetilde{X})_T \mid \widetilde{H} \text{ admissible} \right\},$$

which is a convex cone of random variables in $L^0(\Omega, \mathscr{F}_T, P)$, and

$$\widetilde{C} := \left\{ \widetilde{g} \in L^\infty(\mathscr{F}_T, P) \mid \widetilde{g} \leq \widetilde{f} \text{ for some } \widetilde{f} \in \widetilde{K} \right\}.$$

The condition *no free lunch with vanishing risk (NFLVR)* over $[0,T]$ with respect to $\widetilde{X}$ is

$$\overline{\widetilde{C}} \cap L_+^\infty(\mathscr{F}_T, P) = \{0\},$$

where $\overline{\widetilde{C}}$ denotes the closure of $\widetilde{C}$ with respect to the norm topology of $L^\infty(\mathscr{F}_T, P)$.

For the analogs of Definitions 3.11 and 3.12 in a viable regulated model, simply replace $\widetilde{X}$ with $\widehat{Y}$ and remove all other "$\widetilde{\ }$".

The FTAP states that NFLVR for the integrator of the class of trading strategies is equivalent to the existence of an ELMM for the integrator. In the premodel the integrator is $\tilde{X}$, while in the regulated model it is $\widehat{Y}$. Since $\widehat{Y}$ obeys the SDE (3.5), then from standard theory in order for $(\widehat{Y})_{0\le t\le T}$ to be a local martingale under an equivalent measure Q given by $\frac{dQ}{dP} =: Z_T \in \mathscr{F}_T$, then there exists a strictly positive martingale $(Z_t)_{0\le t\le T}$ satisfying $Z_t = E[Z_T \mid \mathscr{F}_t]$ and having the representation:

$$Z_t := \mathscr{E}(-\theta(Y)\cdot W)_t = \exp\left\{-\int_0^t \theta(Y_s)' dW_s - \frac{1}{2}\int_0^t |\theta(Y_s)'|^2 ds\right\}, \quad 0 \le t \le T, \tag{3.9}$$

where the market price of risk $\theta(\cdot)$ solves the market price of risk equation

$$\sigma(Y_t)\theta(Y_t) = b(Y_t), \quad \text{a.s.}, \; 0 \le t \le T. \tag{3.10}$$

Given an exponential local martingale of the form (3.9), satisfaction of the Novikov criterion,

$$E\left[\exp\left\{\frac{1}{2}\int_0^T |\theta_s|^2\, ds\right\}\right] < \infty, \tag{3.11}$$

is sufficient for implying the martingality of $(Z)_{0\le t\le T}$.

While the NFLVR condition is of theoretical interest, it is not necessarily of practical relevance. If we put ourselves in the situation of having to select from some set of candidate market models, some of which satisfy NFLVR and others of which do not, it may be a hopeless task to figure out whether financial data support or refute NFLVR. In fact, example 4.7 Karatzas and Kardaras (2007) shows that two general semimartingale models on the same stochastic basis may possess the same triple of predictable characteristics, with one admitting an arbitrage while the other does not. Even if we have reason to believe that a model admitting arbitrage or relative arbitrage is an accurate one, it may be the case that the arbitrage portfolios depend in a delicate way on the parameters of the model, b and σ here. In such a case any attempts to estimate these parameters from observed data would likely be too imprecise to lead to an investment strategy that could convincingly be called an approximation to an arbitrage.

In contrast to this, the condition of diversity is supported by world market data and the existence of antitrust laws in developed markets. The condition (3.12) of uniform ellipticity of the covariance is not as readily apparent, but seems to be a reasonable manifestation of the idea that there is always at least some baseline level of volatility in markets. The significance of these two conditions is that in unregulated market models together they imply the existence of a long-only relative arbitrage portfolio that is functionally generated from the market weights (see Section 3.6 herein for the precise conditions as well as Fernholz (1999b); Fernholz and Karatzas (2005, 2009)), not requiring estimation of b or σ. It is therefore of great interest whether or not this implication carries over to regulated markets. The following proposition will be useful in Section 4 for showing that this is not the case.

Proposition 3.13 *If the regulated model is viable, $\widehat{Y}$ satisfies NFLVR over $[0,T]$, and $\sigma(\cdot)$ is bounded on U^x, then any ELMM for $\widehat{Y}$ is an EMM, and no portfolio is a relative arbitrage with respect to any other portfolio over $[0,T]$ in the regulated model.*

Proof To prove the martingality, let measure Q be any ELMM for $\widehat{Y}$, and therefore have the form $\frac{dQ}{dP} = Z_T = \mathscr{E}(-\theta(Y)\cdot W)_T$, where $\theta(Y)$ solves the market price of risk equation (3.10). Then Equation (3.5) implies that

$$d\widehat{Y}_{i,t} = Y_{i,t}\Big(\sum_{\nu=1}^{d} \sigma_{i\nu}(Y_t)dW_{\nu,t}^{(Q)}\Big), \quad 1 \le i \le n, \quad 0 \le t \le T,$$

where

$$W_t^{(Q)} := W_t + \int_0^t \theta(Y_s)ds, \quad 0 \le t \le T$$

is a Q-Brownian motion by Girsanov's theorem. This implies that

$$dV_t^{w,\pi} = V_t^{w,\pi}\pi_t'\sigma(Y_t)dW_t^{(Q)}, \quad 0 \le t \le T.$$

Therefore, $(V^{w,\pi})_{0\le t\le T}$ is an exponential Q-local martingale. Since $Y \in U^x$, $dt \times dP$-a.e., this implies that $\sigma(Y)$ is bounded, $dt \times dP$-a.e. The portfolio π is uniformly bounded by definition, so $(V_t^{w,\pi})_{0\le t\le T}$ is a Q-martingale by the Novikov criterion (3.11).

Now suppose that π is a relative arbitrage with respect to η. Then by $Q \sim P$ it follows that

$$Q(V_T^{w,\pi} \ge V_T^{w,\eta}) = 1 \quad \text{and} \quad Q(V_T^{w,\pi} > V_T^{w,\eta}) > 0.$$

However $(V_t^{w,\pi})_{0\le t\le T}$ and $(V_t^{w,\eta})_{0\le t\le T}$ are both Q-martingales, so their difference is also a Q-martingale, with $E^Q[V_T^{w,\pi} - V_T^{w,\eta}] = w - w = 0$. This contradicts the relative arbitrage property above, so this market admits no pair of relative arbitrage portfolios. □

Some recent investigations pertaining to relative arbitrage include Fernholz et al. (2005), Fernholz and Karatzas (2005), Fernholz and Banner (2008), Ruf (2010), and Mijatović and Urusov (2009), to name a few. An arbitrage is essentially a relative arbitrage with respect to the money market account, modulo the uniform boundedness requirement of portfolios and their prohibition from investing in the money market, both of which can be relaxed as in Fernholz and Karatzas (2010). The existence of a relative arbitrage does not imply the existence of an arbitrage, as illustrated by examples, often called "bubble markets" (see Cox and Hobson (2005); Pal and Protter (2010); Protter et al. (2007)), where there exists an equivalent measure under which the stock process is a strict local martingale. In particular, if $\widetilde{\pi}$ is a relative arbitrage with respect to $\widetilde{\eta}$, then the trading strategy $\widetilde{H} := \widetilde{H}^{1,\widetilde{\pi}} - \widetilde{H}^{1,\widetilde{\eta}}$ need not satisfy the requirement that $\widetilde{H} \cdot \widetilde{X}$ be uniformly bounded from below, so $\widetilde{H}$ need not be admissible.

3.6 Diversity, Intrinsic Volatility, and Relative Arbitrage

The works by Robert Fernholz et al. (Fernholz (1999a, 2002); Fernholz and Karatzas (2005)) on diversity and arbitrage prove that for unregulated markets, over an arbitrary time horizon, there exist strong relative arbitrage portfolios with respect to the market portfolio in any weakly diverse market satisfying certain assumptions and regularity conditions. Furthermore, they show how such relative arbitrages can be constructed as long-only portfolios which are functionally generated from $\widetilde{\mu}$, not requiring knowledge of $\widetilde{b}$ or $\widetilde{\sigma}$. A sufficient set of assumptions and regularity are given by the following.

Assumption 3.1 *(i) The capitalizations are modeled by an Itô process*

$$d\widetilde{X}_{i,t} = \widetilde{X}_{i,t}\left(\widetilde{b}_{i,t}dt + \sum_{\nu=1}^{d}\widetilde{\sigma}_{i\nu,t}dW_{\nu,t}\right), \quad 1 \le i \le n,$$
$$\widetilde{X}_0 = x_0 \in \mathbb{R}^n_{++},$$

where $\widetilde{b}$ and $\widetilde{\sigma}$ are progressively measurable processes satisfying $\forall T \in \mathbb{R}_{++}$,

$$\sum_{i=1}^{n}\left(\int_0^T |\widetilde{b}_{i,t}|dt + \sum_{\nu=1}^{d}\int_0^T \left|\widetilde{\sigma}_{i\nu,t}\right|^2 dt\right) < \infty, \quad a.s.$$

(ii) *The capitalizations' covariance process is uniformly elliptic:*

$$\exists \varepsilon > 0: \ a.s. \quad \varepsilon|\xi|^2 \leq \xi \widetilde{\sigma}_t \widetilde{\sigma}_t' \xi, \quad \forall t \geq 0, \, \forall \xi \in \mathbb{R}^n. \tag{3.12}$$

(iii) *Companies pay no dividends (and therefore can't control their size by this means).*
(iv) *The number of companies is a constant.*
(v) *The market is weakly diverse.*
(vi) *Trading may occur in continuous time, in arbitrary quantities, is frictionless, and does not impact prices.*

These conditions have been generalized in Fernholz and Karatzas (2005). There it is shown that the uniform ellipticity assumption may be relaxed, and the market need not be weakly diverse if it satisfies one of several notions of "sufficient intrinsic volatility." One measure of the intrinsic volatility in the market is the excess growth rate of the market portfolio,

$$\gamma^*_{\widetilde{\mu},t} = \frac{1}{2}\left(\sum_{i=1}^n \widetilde{\mu}_{i,t}\widetilde{a}_{ii,t} - \widetilde{\mu}_t'\widetilde{a}_t\widetilde{\mu}_t\right).$$

The following proposition provides an example of a "sufficient intrinsic volatility" type condition.

Proposition 3.14 (adapted from Proposition 3.1 Fernholz and Karatzas (2005)) *Assume an unregulated market model satisfies items (i), (iii), (iv), and (vi) of Assumption 3.1. Additionally suppose there exists a continuous, strictly increasing function $\widetilde{\Gamma}: [0,\infty) \to [0,\infty)$ with $\widetilde{\Gamma}(0) = 0$, $\widetilde{\Gamma}(\infty) = \infty$, and satisfying a.s.*

$$\widetilde{\Gamma}(t) \leq \int_0^t \widetilde{\gamma}^*_{\widetilde{\mu},s} ds < \infty, \qquad \textit{for all} \quad 0 \leq t < \infty. \tag{3.13}$$

Then there exists a functionally generated, long-only portfolio that is a strong relative arbitrage with respect to the market portfolio over sufficiently long horizon.

Proof See Fernholz and Karatzas (2005).

A diverse regulated market is simply a regulated market in which μ in place of $\widetilde{\mu}$ satisfies Definition 2.3. By Lemma 3.4 of Fernholz and Karatzas (2009) (the proof of which is merely algebraic and has nothing to do with whether the market model is regulated or not) in a uniformly elliptic, diverse market (respectively, regulated market), $\widetilde{\gamma}^*_{\widetilde{\mu}}$ (γ^*_{μ}) satisfies

$$\frac{\varepsilon\delta}{2} \leq \widetilde{\gamma}^*_{\widetilde{\mu},t}, \quad \forall t \geq 0 \qquad \left(\frac{\varepsilon\delta}{2} \leq \gamma^*_{\mu,t}, \quad \forall t \geq 0\right). \tag{3.14}$$

In this equation ε satisfies $\varepsilon|\xi|^2 \leq \xi\widetilde{\sigma}_t\widetilde{\sigma}_t'\xi$ ($\varepsilon|\xi|^2 \leq \xi\sigma_t\sigma_t'\xi$), $\forall t \geq 0$, $\forall \xi \in \mathbb{R}^n$, and δ satisfies $\mu_{(1)} \leq 1-\delta$ ($\widetilde{\mu}_{(1)} \leq 1-\delta$). This implies that in any uniformly elliptic, diverse market (regulated market), that (3.13) (its regulated market counterpart) is satisfied by $\widetilde{\Gamma}(t) = \frac{\varepsilon\delta}{2}t$ ($\Gamma(t) = \frac{\varepsilon\delta}{2}t$). In the examples of Section 4, NFLVR and no relative arbitrage hold for the regulated markets, while diversity and uniform ellipticity also hold, implying that (3.13) is satisfied in these cases. *Therefore, in contrast to the premodel, the conditions of weak diversity and uniform ellipticity together, and thus also the weaker condition of sufficient intrinsic volatility, are not sufficient for the existence of relative arbitrage in the regulated model.*

4 Examples of Regulated Markets

In this section we apply the split-merge regulation of subsection 3.4 to geometric Brownian motion (GBM) and a log-pole market as premodels. In both cases the regulated market is diverse and uniformly elliptic, and therefore satisfies the regulated market analog of the sufficient intrinsic volatility condition (3.13). In both cases the regulated market satisfies NFLVR and admits no pair of relative arbitrage portfolios.

4.1 Geometric Brownian Motion

Consider the case where the unregulated capitalization process is a GBM,

$$d\widetilde{X}_{i,t} = \widetilde{X}_{i,t}\left[b_i dt + \sum_{\nu=1}^{n} \sigma_{i\nu} dW_{\nu,t}\right], \quad 1 \le i \le n$$
$$\widetilde{X}_0 = x_0 \in O^x = \mathbb{R}^n_{++},$$

for some $n \ge 3$, $b \in \mathbb{R}^n$, and $\sigma \in \mathbb{R}^{n\times n}$ of rank n. GBM satisfies NFLVR on all $[0,T]$, $T \in \mathbb{R}_{++}$ and has constant volatility, so it is not weakly diverse on any $[0,T]$ and admits no pair of relative arbitrage portfolios (see Section 6 of Fernholz and Karatzas (2009)). Select $\delta \in (0, \frac{n-1}{n+1})$ and define the regulatory set

$$U^\mu := \{\mu \in \Delta^n_+ \mid \mu_{(1)} < 1-\delta\}.$$

By Lemma 3.10 the associated split-merge rule exists and is viable. Since $\theta := \sigma^{-1}b$ is a constant, the Novikov criterion (3.11) for $Z := \mathscr{E}(-\theta\cdot W)$ is satisfied, and Z is therefore a martingale. This implies that for any $T \in \mathbb{R}_{++}$, Q specified by $\frac{dQ}{dP} := Z_T$ is an ELMM for $(\widehat{Y}_t)_{0\le t\le T}$. Furthermore, $(\widehat{Y}_t)_{0\le t\le T}$ is a Q-martingale, and the regulated market is free of relative arbitrage by Proposition 3.13. The regulated market is diverse since $P(\mu_t \in \bar{U}^\mu,\ \forall t \ge 0) = P(\mu_{(1),t} \le 1-\delta,\ \forall t \ge 0) = 1$, which implies that (3.14) and thus (3.13) are satisfied. Therefore in this regulated market, the notions of sufficient intrinsic volatility and diversity coexist with NFLVR and no relative arbitrage.

4.2 Log-Pole Market

So-called "log-pole" market models provide examples of diverse, unregulated markets. Diversity is maintained in these markets by means of a log-pole-type singularity in the drift of the largest capitalization, diverging to $-\infty$ as the largest weight $\mu_{(1)}$ approaches the diversity cap $1-\delta$. Explicit portfolios which are relative arbitrages with respect to the market portfolio over any prespecified time horizon may be formed by down-weighting the largest company in a controlled manner (see e.g. Fernholz et al. (2005); Fernholz and Karatzas (2009)). This model can be interpreted as a continuous approximation of an economy in which the relative size of the largest company is controlled via a regulator imposing fines on it. When regulatory breakup is applied, keeping the largest weight $\mu_{(1)}$ away from $1-\delta$, then the arbitrage opportunities vanish.

Following Section 9 of Fernholz and Karatzas (2009) (see Fernholz et al. (2005) for more details and generality) fix $n \ge 3$, $\delta \in (0, \frac{1}{2})$ and consider the unregulated capitalization process $\widetilde{X}$, the pathwise unique strong solution to

$$d\widetilde{X}_{i,t} = \widetilde{X}_{i,t}\left(b_i(\widetilde{X}_t)dt + \sum_{\nu=1}^{n} \sigma_{i\nu} W_{\nu,t}\right), \quad 1 \le i \le n,$$
$$\widetilde{X}_0 = x_0 \in O^x := \{x_0 \in \mathbb{R}^n_{++} \mid \mu_{(1)}(x_0) < 1-\delta\},$$

where $\sigma \in \mathbb{R}^{n\times n}$ is rank n. The function $b(\cdot)$ is given by

$$b_i(x) := \frac{1}{2}a_{ii} + g_i 1_{\mathscr{Q}_i^c}(x) - \frac{c}{\delta}\frac{1_{\mathscr{Q}_i}(x)}{\log((1-\delta)/\mu_i(x))}, \qquad 1 \le i \le n,$$

where $\{g_i\}_1^n$ are non-negative numbers, c is a positive number, and

$$\mathscr{Q}_1 := \left\{ x \in \mathbb{R}^n_{++} \mid x_1 \ge \max_{2\le j\le n} x_j \right\}, \qquad \mathscr{Q}_n := \left\{ x \in \mathbb{R}^n_{++} \mid x_n > \max_{1\le j\le m-1} x_j \right\},$$
$$\mathscr{Q}_i := \left\{ x \in \mathbb{R}^n_{++} \mid x_i > \max_{1\le j\le i-1} x_j, \quad x_i \ge \max_{i+1,\le j\le n} x_j \right\}, \quad \text{for } i = 2,\dots,n-1.$$

When $x \in \mathscr{Q}_i$, then x_i is the largest of the $\{x_j\}_1^n$ with ties going to the smaller index. In this model each company behaves like a geometric Brownian motion when it is not the largest. The largest company is repulsed away from the log-pole-type singularity in its drift at $1-\delta$. Strong existence and pathwise uniqueness for this SDE are guaranteed for any x_0 in O^x by Veretennikov (1981) (see also Fernholz et al. (2005)). The capitalizations satisfy $P(\widetilde{X}_t \in O^x,\ \forall t \ge 0) = 1$, so this premodel is diverse. The function $b(\cdot)$ is locally bounded since the coefficients of $1_{\mathscr{Q}_i^c}(x)$ and $1_{\mathscr{Q}_i}(x)$ are continuous on O^x and the singularity at $\mu_{(1)}(x) = 1-\delta$ is away from the boundary of each $\mathscr{Q}_i$ for $\delta \in (0, \frac{1}{2})$. Since the market is diverse and has constant volatility, then by the results of Fernholz (2002); Fernholz and Karatzas (2009) over arbitrary horizon the market admits long-only relative arbitrage portfolios which are functionally generated from the market portfolio. Furthermore since σ is a constant, $\widetilde{X}$ has no ELMM so admits a FLVR.

This model may be regulated in such a way to remove these relative arbitrage opportunities and satisfy NFLVR. Picking $\delta' \in (\delta, \frac{n-1}{n+1})$ and $x_0 \in U^x$, define the regulatory set to be

$$U^\mu := \{\mu \in \Delta^n_+ \mid \mu_{(1)} < 1-\delta'\} \subseteq O^\mu.$$

The associated split-merge regulation rule exists and is viable by Lemma 3.10. The function $b \restriction_{\bar{U}^x} (\cdot)$ is bounded, so taking $\theta(\cdot) := \sigma^{-1}b(\cdot)$, then $\theta(Y)$ is a.s. bounded uniformly in time. This implies that the Novikov criterion (3.11) is satisfied for $Z := \mathscr{E}(-\theta(Y)\cdot X)$, and so Z is a martingale. For any $T \in \mathbb{R}_{++}$, Q specified by $\frac{dQ}{dP} := Z_T$ is an ELMM for $(\widehat{Y}_t)_{0\le t\le T}$. Furthermore $(\widehat{Y}_t)_{0\le t\le T}$ is a Q-martingale, and the regulated market is free of relative arbitrage by Proposition 3.13. The diversity of the regulated market implies that (3.14), and thus (3.13) are satisfied. Therefore in this regulated market, the notions of sufficient intrinsic volatility and diversity coexist with NFLVR and no relative arbitrage.

The pathology of this premodel is that the largest company's drift approaches $-\infty$ as $\mu_{(1)}$ approaches $1-\delta$. The cure is to prevent the largest company from approaching $1-\delta$ by regulation and thus bound the worst expected rate of return. The pathological region of Δ^n_+ is removed from μ's state space by the regulation procedure, and the result is an arbitrage-free market.

5 Conclusions

Models in which diversity is maintained by a drift-type condition, whereby the rate of expected return of the largest company must become unboundedly negative compared to the rate of expected return of some other company in the economy, cover only one particular mechanism by which diversity may be achieved. These are reasonable models for markets in which diversity is maintained by some combination of fines on big companies imposed by antitrust regulators, and/or the biggest company consistently delivering less return than the other companies for other reasons. In such markets there is an intuitive undesirability in holding

the stock of the largest company, since its upside potential is limited relative to that of the other companies. Fernholz showed that this is not merely a vague undesirability, but that any passive portfolio holding shares of the biggest company can be strictly outperformed by functionally generated portfolios which are relative arbitrages with respect to the former.

If regulators maintain diversity within an equity market by utilizing regulatory breakup, then the situation is quite different. This mechanism need not open the door to arbitrage. It entails no systematic debasement of the total capital in the economy, and, for many models, can be shown to be arbitrage-free, admitting an equivalent martingale measure.

The current situation in U.S. markets is that regulatory breakups are uncommonly used, and primarily in cases reversing provisionally approved mergers. This suggests that the previous conclusion of Fernholz, Karatzas et al. in Fernholz and Karatzas (2005) that in the past conditions in U.S. markets have likely been compatible with functionally generated relative arbitrage with respect to the market portfolio, is not threatened by this result. If, however, regulatory breakup were to become a primary tool of antitrust regulators, then, modulo our assumption of portfolio wealth conservation, the argument for existence of functionally generated relative arbitrage in diverse markets would be substantially weakened.

The notions of diversity combined with uniform ellipticity, and the more general "sufficient intrinsic volatility of the market" are useful conditions in that they can be tested by empirical observations. This is in contrast to the rather abstract and normative condition of existence of an equivalent martingale measure, for which it may be hopeless to make a case for or against via observed data alone. That these conditions do not imply relative arbitrage in regulated market models prompts the question of whether a general, empirically verifiable condition can be found that implies relative arbitrage for both regulated and unregulated market models.

6 Proofs

Proof (Proof of Lemma 3.9) For $\mu_t := \mu(Y_t)$, let $G_t := G(\mu_t),\ \forall(\omega,t) \in [0,\tau_\infty)$. By Definition 3.4 of $\widehat{Y}$ and 3.5, we can decompose $G_{t\wedge\tau_k}$ as

$$G_{t\wedge\tau_k} = G_0 + \sum_{m=1}^{k} \int_{t\wedge\tau_{m-1}^+}^{t\wedge\tau_m} dG_t + \sum_{m=1}^{N_t\wedge(k-1)} \left[G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}\right]. \tag{6.1}$$

On $(\tau_{k-1},\tau_k]$ by Itô's formula, the process μ obeys

$$\begin{aligned} d\mu_{i,t} &= \mu_{i,t}\Big[\Big((b_i(X_t) - \sum_{j=1}^{n} a_{ij}(X_t)\mu_{j,t} - [\mu_t' b(X_t) - \mu_t' a(X_t)\mu_t]\Big)dt \\ &\quad + \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(X_t) - [\mu_t'\sigma(X_t)]_\nu\big)\, dW_{\nu,t}\Big], \\ &= B_{i,t}dt + \sum_{\nu=1}^{d} R_{i\nu,t} dW_{\nu,t}, \quad 1 \le i \le n. \end{aligned}$$

The processes B and R are bounded on $(0,\tau_\infty)$, since $b(\cdot)$ and $\sigma(\cdot)$ are uniformly bounded on U^x. Defining $\widehat{G}_t := G_t - \sum_{m=1}^{N_t}[G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}],\ \forall(t,\omega) \in [0,\tau_\infty)$, then by Itô's formula $\widehat{G}$ is an Itô process on $[0,\tau_\infty)$, and so there exist processes C and S taking values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$, respectively, such that

$$d\widehat{G}_t = C_t dt + S_t dW_t, \quad \text{on } (0,\tau_\infty).$$

The integrands C and S are uniformly bounded on $(0,\tau_\infty)$ since the first and second derivatives of $G(\cdot)$ are by assumption bounded on Δ_+^n, and B, R above are uniformly bounded on $(0,\tau_\infty)$. This implies that $\int_0^{t\wedge\tau_\infty} C_s ds$ and $\int_0^{t\wedge\tau_\infty} S_s dW_s$ are well-defined for all $t>0$ by the theories of Lebesgue and stochastic integration. Therefore $\lim_{k\to\infty}(\mathbf{1}_{\{\tau_\infty<\infty\}}\widehat{G}_{\tau_k})\in\mathbb{R}$ a.s.

By (6.1) and the definition of $\widehat{G}$, we have:

$$G_{\tau_k} = \widehat{G}_{\tau_k} + \sum_{m=1}^{k-1}\left[G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}\right]. \tag{6.2}$$

On $\{\tau_\infty<\infty\}$ by assumption either

$$\sum_{m=1}^{k-1}\left[G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}\right] \underset{k\to\infty}{\longrightarrow} \infty, \quad \text{a.s.},$$

or

$$\sum_{m=1}^{k-1}\left[G(\mathfrak{R}^\mu(\mu_{\tau_m})) - G_{\tau_m}\right] \underset{k\to\infty}{\longrightarrow} -\infty, \quad \text{a.s.}$$

But in (6.2) $\{\widehat{G}_{\tau_k}\}_1^\infty$ converges in $\mathbb{R}$ a.s. on $\{\tau_\infty<\infty\}$, and $G(\cdot)$ is a bounded function by assumption, so (6.2) implies that $P(\tau_\infty<\infty)=0$. □

Proof (Proof of Lemma 3.10) Fix $n\geq 3$ and $\delta\in(0,\frac{n-1}{n+1})$. The boundary of U^μ in Δ_+^n is

$$\partial U^\mu = \{\mu\in\Delta_+^n \mid \mu_{(1)} = 1-\delta\}.$$

The set U^μ is non-empty and open, and by assumption satisfies $U^\mu\subseteq O^\mu$. To check that $\breve{\mathfrak{R}}^\mu$ is into $\bar{U}^\mu$, note that $\mu\in\partial U^\mu \Rightarrow \mu_{(1)} = 1-\delta \Rightarrow \mu_{(n)}+\mu_{(n-1)} \leq \frac{2\delta}{n-1} < \frac{2}{n+1} < 1-\delta$, where the first inequality follows from $\sum_{j=1}^n \mu_j - \mu_{(1)} = \delta$, implying that the smallest two weights can sum to at most $\frac{2\delta}{n-1}$, and the second inequality follows from $\delta\in(0,\frac{n-1}{n+1})$. This implies that all of the “new companies” created by $\breve{\mathfrak{R}}$ are of relative size strictly smaller than $1-\delta$. So $[\breve{\mathfrak{R}}(\mu)]_{(1)}\leq 1-\delta$ which implies that $\breve{\mathfrak{R}}$ is into $\bar{U}^\mu$. If there were k companies of relative size $1-\delta$ for $\mu\in\partial U^\mu$, then $\breve{\mathfrak{R}}^\mu(\mu)$ has $k-1$ companies of relative size $1-\delta$. Therefore, applying the n-fold composition $(\breve{\mathfrak{R}}^\mu\circ\ldots\circ\breve{\mathfrak{R}}^\mu)$ to $\mu\in\bar{U}^\mu$ results in no companies of relative size $1-\delta$. This implies that $\mathfrak{R}^\mu$ of Definition 3.8 is into U^μ, making $(U^\mu,\mathfrak{R}^\mu)$ a regulation rule and therefore a split-merge rule.

Consider the entropy function

$$\begin{aligned} S &: \mathbb{R}_{++}^n \to \mathbb{R}, \\ S(x) &= -\sum_{i=1}^n x_i \log x_i. \end{aligned}$$

We examine the change in entropy resulting from $\breve{\mathfrak{R}}$. For $\mu\in\partial U^\mu$ we have $\mu_{(1)}=1-\delta$, and so

$$\begin{aligned} S(\breve{\mathfrak{R}}^\mu(\mu)) - S(\mu) &= -\left[2\frac{\mu_{(1)}}{2}\log(\frac{\mu_{(1)}}{2}) + (\mu_{(n)}+\mu_{(n-1)})\log(\mu_{(n)}+\mu_{(n-1)})\right] \\ &\quad + \left[\mu_{(1)}\log\mu_{(1)} + \mu_{(n)}\log\mu_{(n)} + \mu_{(n-1)}\log\mu_{(n-1)}\right], \\ &= (1-\delta)\log 2 - (\mu_{(n)}+\mu_{(n-1)})\log(\mu_{(n)}+\mu_{(n-1)}) \\ &\quad + 2\left(\frac{\mu_{(n)}\log\mu_{(n)} + \mu_{(n-1)}\log\mu_{(n-1)}}{2}\right). \end{aligned}$$

Applying Jensen's inequality to the convex function $x \mapsto x \log x$, we get

$$\begin{aligned} S(\check{\mathfrak{R}}^{\mu}(\mu)) - S(\mu) \geq (1-\delta)\log 2 \\ &+ (\mu_{(n)} + \mu_{(n-1)}) \left[-\log(\mu_{(n)} + \mu_{(n-1)}) + \log(\frac{\mu_{(n)} + \mu_{(n-1)}}{2}) \right], \\ &= (1-\delta)\log 2 - (\mu_{(n)} + \mu_{(n-1)})\log 2, \\ &\geq \log 2 \left[1 - \delta - \frac{2\delta}{n-1} \right] > 0, \end{aligned}$$

where the second to last inequality follows from the fact that $\sum_{j=1}^{n} \mu_j - \mu_{(1)} = \delta$, so the smallest two weights can sum to at most $\frac{2\delta}{n-1}$. The last inequality follows from the supposition that $\delta \in (0, \frac{n-1}{n+1})$. From this, the change in entropy of $\mathfrak{R}$ can be seen to satisfy

$$S(\mathfrak{R}^{\mu}(\mu)) - S(\mu) \geq \left[1 - \delta \left(\frac{n+1}{n-1} \right) \right] \log 2 > 0, \quad \forall \mu \in \partial U^{\mu}.$$

For $\varepsilon \in \mathbb{R}_{++}$, we may define the shifted entropy function

$$\begin{aligned} S^{(\varepsilon)} &: \Delta^n_+ \to \mathbb{R} \\ S^{(\varepsilon)}(\mu) &:= S(\varepsilon \mathbf{1}_n + \mu) = -\sum_{i=1}^{n} (\mu_i + \varepsilon)\log(\mu_i + \varepsilon), \end{aligned}$$

where $\mathbf{1}_n$ is the column vector of n ones. For any $\kappa \in (0, \infty)$, the entropy function S restricted to domain $\{\mu : 0 < \mu_i \leq \kappa, \text{ for } 1 \leq i \leq n\}$ is uniformly continuous, so therefore $\varepsilon \in (0,1)$ can be chosen such that

$$\inf \left\{ S^{(\varepsilon)}(\mathfrak{R}^{\mu}(\mu)) - S^{(\varepsilon)}(\mu) \mid \mu \in \partial U^{\mu} \right\} > 0. \tag{6.3}$$

The shifted entropy function satisfies $S^{(\varepsilon)} \in C_b^2(\Delta^n_+, \mathbb{R})$, so for an SDE (2.1) with $b(\cdot)$ and $\sigma(\cdot)$ bounded on U^x, an application of Lemma 3.9 with $G = S^{(\varepsilon)}$ proves the viability of $(U, \mathfrak{R})$. □

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References

Cox, A.M.G., Hobson, D.G.: Local martingales, bubbles, and option prices. Finance and Stochastics **9**, 477–492 (2005)

Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Mathematische Annalen **300**, 463–520 (1994)

Delbaen, F., Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. Mathematische Annalen **312**, 215–250 (1998)

Delbaen, F., Schachermayer, W.: The Mathematics of Arbitrage. First edn. Springer, Berlin (2006)

European Commission v Intel: Antitrust: Commission imposes fine of €1.06 bn on Intel for abuse of dominant position; orders Intel to cease illegal practices. `http://europa.eu/rapid/pressReleasesAction.do?reference=IP/09/745&format=HTML&aged=0&language=EN&guiLanguage=en` (2009)

Fernholz, D., Banner, A.: Short-term relative arbitrage in volatility-stabilized markets. Annals of Finance **4**, 445–454 (2008)

Fernholz, D., Karatzas, I.: On optimal arbitrage. Annals of Applied Probability **20**, 1179–1204 (2010)

Fernholz, E.R.: Antitrust and the no-arbitrage hypothesis. Tech. rep., INTECH (1998). `https://ww3.intechjanus.com/Janus/Intech/intech?command=intechSecureFileController&assetName=INTECHResearch_AntitrustNoArbitrageDoc`

Fernholz, E.R.: On the diversity of equity markets. Journal of Mathematical Economics **31**, 393–417 (1999a)

Fernholz, E.R.: Portfolio generating functions. In: Avellaneda, M. (ed.) Quantitative Analysis in Financial Markets, pp. 344–364. World Scientific, Singapore (1999b)

Fernholz, E.R.: Stochastic Portfolio Theory. First edn. Springer, Berlin (2002)

Fernholz, E.R., Karatzas, I.: Relative arbitrage in volatility-stabilized markets. Annals of Finance **1**, 149–177 (2005)

Fernholz, E.R., Karatzas, I.: Stochastic portfolio theory: An overview. In: Bensoussan, A., Zhang, Q. (eds.) Handbook of Numerical Analysis: Volume XV: Mathematical Modeling and Numerical Methods in Finance, pp. 89–167. North Holland, Oxford (2009)

Fernholz, E.R., Karatzas, I., Kardaras, C.: Diversity and relative arbitrage in equity markets. Finance and Stochastics **9**, 1–27 (2005)

Karatzas, I., Kardaras, C.: The numéraire portfolio in semimartingale financial models. Finance and Stochastics **11**, 447–493 (2007)

Karatzas, I., Shreve, S.E.: Methods of Mathematical Finance. First edn. Springer, Berlin (1998)

Microsoft Corp. v European Commission: Microsoft Corp. v Commission of the European Communities. Case T-201/04). `http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=CELEX:62004A0201:EN:NOT` (2004)

Mijatović, A., Urusov, M.: Deterministic criteria for the abscence of arbitrage in diffusion models. Finance and Stochastics (2009). To Appear. Preprint. `http://arxiv.org/abs/1005.1861`

Osterrieder, J.R., Rheinländer, T.: Arbitrage opportunities in diverse markets via a non-equivalent measure change. Annals of Finance **2**, 287–301 (2006)

Pal, S., Protter, P.: Analysis of continuous strict local martingales via h-transforms. Stochastic Processes and their Applications (2010). To Appear

Protter, P., Jarrow, R.A., Shimbo, K.: Asset price bubbles in complete markets. In: Fu, M.C., Jarrow, R.A., Yen, J.Y.J., Elliott, R.J. (eds.) Advances in Mathematical Finance, pp. 97–121. Birkhäuser, Boston (2007)

Ruf, J.: Optimal trading strategies under arbitrage (2010). Preprint. `http://arxiv.org/abs/1003.4797v1`

Standard Oil v United States: *The Standard Oil Company of New Jersey, et al v. The United States*, 221 U.S. 1. `http://laws.findlaw.com/us/221/1.html` (1911)

United States v AT&T: *United States v. AT&T*, 552 F.Supp. 131 (DDC 1982) (1982)

Veretennikov, A.Y.: On strong solutions and explicit formulas for solutions of stochastic integral equations. Sbornik: Mathematics **39**, 387–403 (1981)